Comment on "Periodic phase synchronization in coupled chaotic oscillators"

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Kye *et al.* [Phys. Rev. E **68**, 025201 (2003)] have recently claimed that, before the onset of chaotic phase synchronization in coupled phase coherent oscillators, there exists a temporally coherent state called periodic phase synchronization (PPS). Here we give evidence that some of their numerical calculations are flawed, while we provide theoretical arguments that indicate that PPS is not to be expected generically in this type of systems.

INTRODUCTION

In [1], Kye *et al.* claim that as a part of the route to chaotic phase synchronization for unidirectionally and bidirectionally coupled Rössler oscillators, there exists a temporally organized state, which they call *periodic phase synchronization* (PPS), just before the onset of chaotic phase synchronization (CPS). This state would be characterized by a maximal coherence of the temporary phase locking (TPL) time τ (=the time between consecutive phase slips). They also report that the Lyapunov exponents (LE's) of the system are an indicator allowing one to characterize PPS. Although they only analyze the Rössler system, it is implied that the phenomenon is generic and happens in the route to CPS in coupled chaotic oscillators (at least if the oscillators are phase coherent). We report serious numerical flaws in their published results and also that one should not expect generically a temporally coherent state as a part of the transition to CPS.

The authors of [1] claim, by calculating the coherence measure $P(\varepsilon) = \sqrt{\text{var}(\tau)}/\langle\tau\rangle$, that a minimum of P occurs before the onset of phase synchronization. We have recalculated this quantity for the two situations studied in [1]: namely, unidirectional [2] and bidirectional coupling, Figs. 3 and 4(b) in Ref. [1]. We find quite different results, even in qualitative terms.

Our results, Figs. 1 and 2, indicate that the dependence of P on ε is rather flat, until the threshold of CPS is approached. Close to the critical coupling $\varepsilon \to \varepsilon_{PS}$, P approaches 1, because τ is large and phase jumps become uncorrelated, leading to an exponential distribution of τ and, consequently, $\langle\tau\rangle = \sqrt{\text{var}(\tau)}$. Surprisingly, in [1], P approaches 1 also for $\varepsilon \to 0$, which is strongly counterintuitive (we are dealing with weakly nonisochronous oscillators) and totally different from our numerical simulations, Figs. 1 and 2. We suspect that this discrepancy is due to the method of detecting phase slips illustrated in Fig. 2 of [1]. In our computations, we have directly measured the phases of the oscillators to detect phase slips.

To study *generic* features of the transition to CPS, one can rely on a general model (the "special flow") based on a Poincaré map (at a fixed phase of the oscillator) [3]. It describes a *weakly* periodically forced chaotic oscillator. The amplitude of the oscillator (x) and the phase of the external force (ψ) can be modeled by a two-dimensional chaotic map:

$$x_{n+1} = f(x_n), \tag{1}$$

$$\psi_{n+1} = \psi_n + T(x_n) + \Phi(\psi_n, x_n). \tag{2}$$

The dynamical system (1) is assumed to exhibit chaotic behavior, and the function f can be considered (in first approximation) to depend only in x because at low coupling only the phase, not the amplitude, is affected.

A particular example is [3]

$$x_{n+1} = 1 - 2|x_n|, \tag{3}$$

$$\psi_{n+1} = \psi_n + \nu(T_0 + \delta x_n) + \varepsilon \cos(\psi_n), \tag{4}$$

where ν accounts for the detuning between the forcing and oscillator periods and ε is the coupling strength. The parameter δ determines the nonisochronicity of the oscillator—i.e., the amplitude dependence of the period. More complicated dependences on the amplitude, $|\delta x_n|$ and $(\delta x_n)^2$, were tested, finding no qualitative difference.
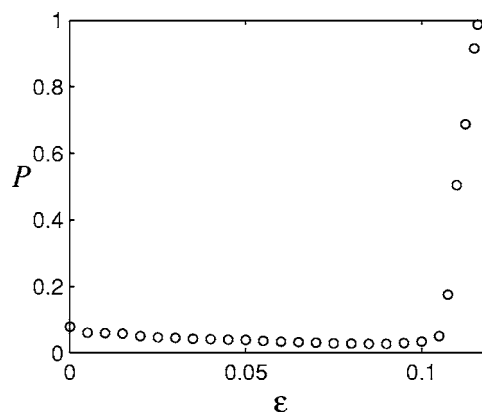

FIG. 1. Measure of P as a function of ε, for a Rössler oscillator forced by another one. Equivalent to Fig. 3 in [1].

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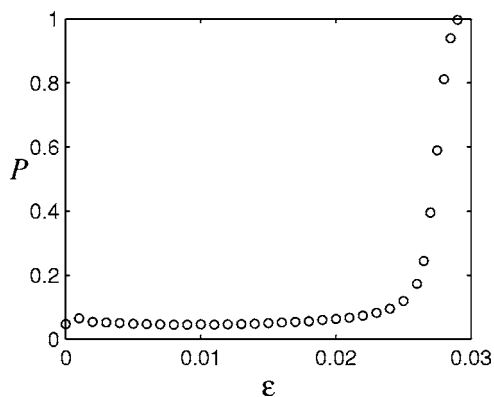†Electronic address: manuel@imedea.uib.es

FIG. 2. Measure of P as a function of ε, for two coupled Rössler oscillators. Equivalent to Fig. 4(b) in [1].

FIG. 3. Coherence factor and Lyapunov exponent for the forced map, Eq. (4), with $\nu=0.98$, $\delta=0.1$, and $T_0=2\pi$.

For typical values of the parameters (see Fig. 3), the behavior of the coherence factor P is always monotonically increasing: Rather flat for small coupling and close to 1 near the onset of CPS.

In addition to the coherence measure P, in Ref. [1] it is also argued that one of the vanishing LE's becomes negative in a short interval (a "dip" following the terminology in [1]) at around the same value of ε at which P is supposed to have a minimum. It may be seen in Fig. 3 that the model in Eqs. (3) and (4) exhibits a monotonically decreasing Lyapunov exponent λ (associated with the dynamics of ψ). This behavior can be expected to be typical [4].

The model in Eqs. (1) and (2) is intended as a zeroth-order approximation. A more realistic implementation would include an additional term depending on the phase in Eq. (1); i.e., the amplitude is not totally insensitive to the phase (see [5]). Further, most chaotic attractors are not hyperbolic, and thus the Rössler system exhibits crises, under parameter variation, that give rise to periodic windows and banded attractors. The logistic map is therefore more suited than the tent map in Eq. (1) to study the effect of nonhyperbolicity. Hence, these effects could be taken into account substituting Eq. (3) by $x_{n+1}=ax_n(1-x_n)+\varepsilon\rho\sin(\psi_n)$. The result of this implementation is that for some parameter values, particularly when a is chosen close to a periodic window (and provided that ρ is large enough), the second LE may become negative in a short interval. But it is noteworthy that parameter P is rather insensitive to the fluctuation of the LE.

In accordance with the special flow model, we emphasize that although a dip is observed for the system of coupled
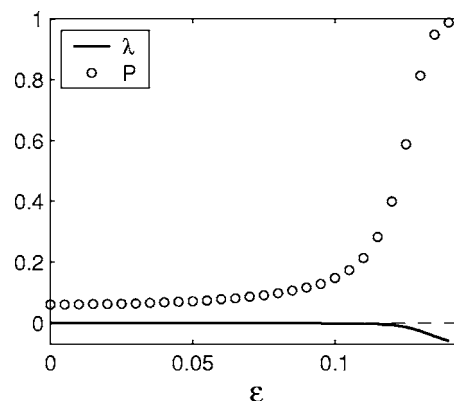
oscillators [1,6] around $\varepsilon=0.023$, P remains monotonically increasing (Fig. 2). When the coupling is unidirectional, there is a stronger distortion in the topology of the driven oscillator, which can be seen, e.g., in the fact that a large value of ε is needed in order to achieve phase synchronization. A somewhat unusual set of parameters has been chosen in [1]; in addition to the detuning in the parameters $\omega_{1,2}$ that control the natural frequencies, the coefficient of the linear y term in $\dot{y}$ is different in the two oscillators: 0.15 and 0.165, respectively. Nonetheless, P continues to exhibit quite a flat dependence on ε until CPS is approached, Fig. 1.

In summary, we have given reasons to support the conclusion that there is no evidence that the reported periodic phase synchronization behavior occurs before the onset of phase synchronization in coupled phase-coherent chaotic oscillators, as reported in Ref. [1]. Recently, PPS has been reported in an experiment with a periodically forced laser [7]. In this system, P presents a minimum, but with a value not far from 1:0.7. As long as lasers exhibit chaotic attractors of Shilnikov type, which are known to be strongly noncoherent, values of P near 1 are not surprising. We are of the opinion that the results of this experiment [7] (obtained with a noncoherent attractor) cannot be extrapolated to coherent attractors, contrary to what is argued by the authors of [1]. Conversely, the results with coherent attractors cannot be claimed in support of PPS for homoclinic attractors.

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